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‘AdS₅’ Geometry Beyond Space-time and 4D Noncommutative Space-time

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Abstract : We discuss a 4D noncommutative space-time as suggested by the version of quantum (deformed) relativity which provides a classical geometry picture as an ‘AdS₅’. The 4D noncommutative space-time is more like a part of a phase space description, in accordance with the quantum notion – quantum mechanics talks about only states but not configurations. The ‘AdS₅’ picture also illustrates the classical 4D space-time is to be described as part of a bigger geometry beyond space-time at the quantum level. The radically new picture of quantum ‘space-time’ is expected to provide the basis for a (still to be formulated) new approach to quantum gravity with fundamental constants (quantum) hbar and Newton’s constant G put at a similar level as c, the speed of light.

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1 Introduction

We discuss some aspect of our new approach to think about possible formulation of the the physics of ‘quantum space-time’ or ‘quantum gravity’. Our approach is based on some basic background perspective. We consider there being a true microscopic/quantum structure of space-time itself and seek a direct description of that, without going through a scheme of quantization. The latter is in accordance of various attempts to get to a foundation of quantum physics with some sort of ‘hidden variable theory’, except that we are ready to take the notion much beyond the framework of the usual (classical) geometry for space-time. It is our belief that new conceptual thinking about what is ‘space-time’ is necessary to resolve all the issues. An example of a theory in a similar spirit is offered by the (Matrix Model) Trace Mechanics published by Adler[1]. It is a new fundamental dynamics formulated on a matrix model geometry, a typical class of noncommutative geometry. We do believe the generic noncommutative geometry, as the natural mathematical generalization of the classical/commutative geometry [2], should be the right mathematical setting. In more exact terms, we believe, or postulate that,

$$\begin{aligned} & \text{\it Non-Commutative Geometry is to Quantum Gravity} \\ & \text{\it as Non-Euclidean Geometry is to Classical (Einstein) Gravity.} \end{aligned}$$

While Adler starts by assuming a particular noncommutative geometry to be the right space-time background to formulate the fundamental theory behind quantum mechanics, we want to find a guiding principle on how to think about the ‘space-time’ structure itself. A plausible answer is given by the principle of relativity symmetry stabilization [3].

From a pure mathematical point of view, stabilization gives a new symmetry that has the old one as a contraction limit. The Lorentz symmetry is exactly a stabilization/deformation of the Galilean counterpart, with the deformation parameter given by $1/c^2$. Philosophically, one can argue that only stable symmetries can be scientifically verified to be correct. The contracted unstable limit is a singular point on the ‘parameter space’, requiring infinite experimental precision to be confirmed against the stable symmetry which admits no other parameter value. The pioneering work of Snyder in 1947[4] had initiated the idea of a symmetry deformation being necessary to implement an invariant quantum scale. The idea has been gaining more attention since the turn of the century [5].

A ‘quantum space-time’ to be directly described, without going through any quantization procedure, will have its own relativity. As suggested by Snyder, simple physics consideration can help identifying the deformed relativity symmetry. The deformations could be nicely formulated as Lie algebra stabilizations [3]. Following the line of thinking, we implemented in Ref.[6] a linear realization perspective. The linear realization scheme, applied to the Einstein relativity as the deformation of Galilean relativity on 3D space, is nothing other than the 4D

Minkowski space-time picture. Such a mathematically conservative approach, however, leads to a very radical physics perspective, that space-time is to be described as part of something bigger [6], what we called the quantum world in Ref.[7]. In the latter paper we identify the quantum relativity symmetry as $SO(2, 4)$, with a linear realization on a 6D classical geometry beyond space-time providing a description of a 4D noncommutative (quantum) space-time. The quantum world is really the coset space $SO(2, 4)/SO(2, 3)$, *i.e.* the hypersurface $\eta_{MN}X^M X^N = -1$ [$\eta_{MN} = (1, -1, -1, -1, -1, 1)$]. For the lack of a better terminology, we denote it by ‘AdS₅’¹. With the $SO(2, 4)$ relativity, fundamental constants \hbar and G are essentially putting on exactly the same footing as c .

2 Getting to the relativity symmetry $SO(2, 4)$

In summary, we have the stabilizations and extensions by translations sequence :

$$\begin{aligned} ISO(3) &\rightarrow SO(1, 3) \hookrightarrow ISO(1, 3) \\ &\rightarrow SO(1, 4) \hookrightarrow ISO(1, 4) \rightarrow SO(2, 4) \end{aligned}$$

The ***ISO(3)*** algebra is unstable, with ***SO(1, 3)*** a possible stabilization result. The only other mathematical option is ***SO(4)***. It is the physics picture of the deformation parameter being identified as essentially an upper limit of admissible speed that fix the right choice. A linear realization adopts the 4D Minkowski space-time M^4 over that of 3D space \mathbb{R}^3 as the arena for the relativity theory; ***SO(1, 3)*** is the isometry of M^4 . With the linear realization comes the natural extension of the Lorentz symmetry to Poincaré symmetry ***SO(1, 3)***. The latter is again an unstable symmetry, to be stabilized to ***SO(1, 4)*** with an energy-momentum invariant, the Planck momentum κc . The physics behind the stabilization step as well as the last one is summarized in table 1. In the case of ct in going from \mathbb{R}^3 to M^4 , a linear realization asks to have a fifth coordinate $\kappa c \sigma$ be incorporated to form the arena for the ***SO(1, 4)*** relativity. The σ -coordinate is quite peculiar. It has a space-time geometric signature and a physics dimension of ($\frac{\text{time}}{\text{mass}}$). It is neither space or time. After all, whether the extra coordinate possibly has the physics meaning of a space-time one is a question of how one formulate the theory. For example, taking the metric of the 5D geometry as a gravitational field will be giving σ a space-time (indeed space rather than time) interpretation. The latter kind of ready to be formulated theories will

¹Note that a version of the table published earlier[7] has mistakenly put down the coset space characterized by $\eta_{MN}X^M X^N = -1$ here as $SO(2, 4)/SO(1, 4)$. The careless mistake was propagated to the talk presentation file at the SMP 40 conference. The possible inconsistence was first brought to the attention of the author by the group headed by H.Y. Guo at the Institute of Theoretical Physics, CAS, Beijing. In relation to that, some physicists may consider the term AdS₅ inappropriate to be used to named the coset space. We put the term in quotes, to alert readers and avoid unnecessary confusion.

Table 1: The Three Deformations Summarized:

$\Delta x^i(t) = v^i \cdot t$	$\Delta x^\mu(\sigma) = p^\mu \cdot \sigma$	$\Delta x^A(\rho) = z^A \cdot \rho$
$ v^i \leq c$ $-\eta_{ij}v^i v^j = c^2 \left(1 - \frac{1}{\gamma^2}\right)$	$ p^\mu \leq \kappa c$ $\eta_{\mu\nu}p^\mu p^\nu = \kappa^2 c^2 \left(1 - \frac{1}{\Gamma^2}\right)$	$ z^A \leq i\ell$ $\eta_{AB}z^A z^B = -\ell^2 \left(1 + \frac{1}{G^2}\right)$
$M_{oi} \equiv N_i \sim P_i$ $[N_i, N_j] \longrightarrow -i M_{ij}$	$J_{\mu 4} \equiv O_\mu \sim P_\mu$ $[O_\mu, O_\nu] \longrightarrow i M_{\mu\nu}$	$J_{A5} \equiv O'_A \sim P_A$ $[O'_A, O'_B] \longrightarrow i J_{AB}$
$\bar{u}^4 = \frac{\gamma}{c}(c, v^i)$ $\eta_{\mu\nu}u^\mu u^\nu = 1$ $\mathbb{R}^3 \rightarrow SO(1, 3)/SO(3)$	$\bar{\pi}^5 = \frac{\Gamma}{\kappa c}(p^\mu, \kappa c)$ $\eta_{AB}\pi^A \pi^B = -1$ $M^4 \rightarrow SO(1, 4)/SO(1, 3)$	$\bar{X}^6 = \frac{G}{\ell}(z^A, \ell)$ $\eta_{MN}X^M X^N = -1$ $M^5 \rightarrow SO(2, 4)/SO(2, 3)$

be in conflict with the role of σ as seen from the relativity symmetry stabilization physics, or equivalently for having the right Einstein limit. The linear realization does suggest a radical departure of our physics thinking. It provokes even the question of if we could still formulate dynamics in the way we used to. Anyway, we have again a natural extension of the symmetry to **$ISO(1, 4)$** , and again a stabilization awaits. It may look like such a stabilization followed by extension sequence will go on forever. However, as illustrated in table 1, we take the last stabilization as corresponding a ‘length’ invariant, as terminate the sequence. This is like implementing a Planck length independent of the Planck mass or momentum. The idea is that the quantum scale as usually described equally effectively by Planck mass κ and Planck length ℓ assumes \hbar . To retrieve \hbar from the symmetry stabilization picture similar to c , one should rather use both Planck momentum and Planck length for the two deformations following the familiar one with c , getting \hbar from the identity $\hbar = \kappa c \ell$. The c constraint enforces the velocity space to be the coset **$SO(1, 3)/SO(3)$** . Likely, the second deformation curves the momentum space and the last curves the ‘space-time’ arena itself. Hence, though the quantum relativity symmetry **$SO(2, 4)$** is to be linearly realized as an isometry of a 6D (beyond space-time) geometry, the relevant quantum world is only the hypersurface given by $\eta_{MN}X^M X^N = -1$. Coordinate translations are simply not admissible symmetries.

For the readers who are not familiar with the idea of symmetry stabilization, we present here a simply argument for an easy appreciation of the **$ISO(3)$** to **$SO(1, 3)$** story. Firstly, note that the only difference in the two algebras is the commutator of two velocity boosts. Commutators of Lorentz boost generators are rotation generators, as given by

$$[N_i, N_j] = -i \frac{1}{c^2} M_{ij},$$

where $\frac{1}{c^2}$ has been explicitly shown. The commuting algebra of Galilean boosts

is retrieved at the $\frac{1}{c^2}$ goes to zero limit. The latter is unstable, as taking any small change in value of the zero structural constant changes the algebra. The Lorentz algebra is stable; different values of c give isomorphic algebra connected by a simple scaling. A more direct way is see it is to realize that the mathematics of symmetry algebra sees no units in physics. The value of c depends, though, on our choice of units, we can make it 3×10^{-7} (km ps $^{-1}$) or 10^{28} (A yr $^{-1}$). The symmetry of space-time is of course independent of what units physicists cooked up to measure things inside. The value of c tells only when we would be in a regime where the Galilean symmetry, as an approximation of the Lorentz symmetry, is good enough to describe physics.

3 The new physics picture

We have the quantum relativity symmetry obtained through the Lie algebra stabilization scheme with quite limited physics inputs. To really construct a theory to be tested experimentally, we need to take it beyond kinematics. The radical beyond space-time picture posts a daunting challenge, as we have to figure out the role of the new σ and ρ coordinates in any picture of dynamics. At the most primitive level, dynamics is a study of motion and motion is characterized by change of spatial position with respect to time. The σ and ρ coordinates would have apparently nothing to do with motion then. Obviously, understanding the physics role of σ and ρ will be a key to confront the theoretical challenge ahead.

The symmetry deformation scheme does tell us something about σ and ρ . In fact, hidden in the mathematics of table 1 are the sort of definitions for the quantities. As $dt = \frac{dx}{v}$, we can start thinking about σ through $p^\mu = \frac{dx^\mu}{d\sigma}$. This is actually another radical departure from the conventional mechanics. It is nothing less than a new definition of the energy-momentum four-vector. It has been argued in Ref.[6] that this is admissible so long as the physics theory guarantees $p^\mu = mcu^\mu$, the Einstein energy-momentum at the proper limit. From the latter requirement, one retrieves for such cases $\sigma = \frac{\tau}{m}$, the proper time over the rest mass of an Einstein particle. That is very interesting for a coordinate with a space-like geometric signature indeed. Some aspects of the new relativity symmetry transformations involving σ , we called momentum boosts, have been discussed in Ref.[6]. It may characterize some transformations to quantum frames of reference, with features in basic correspondence with what has been discussed by some authors of the topic [8, 9]. It is interesting to note that parameter essentially the same as σ has been used quite lot in various approaches to (Einstein) relativistic quantum mechanics in somewhat ambiguous ways. Our new relativity picture may have to put some of that on solid theoretical footing, and hence retrieve a better understanding about the σ coordinate [10]. The ρ -coordinate looks simpler. The translational boosts, transformations involving ρ are like effective translations on the quantum world,

liable to be interpreted as a change in coordinate of the geometric description. The set of z^A 's actually serves as Beltrami-type coordinate system [7, 11].

Both the σ and ρ can also be shown to be closely related to the idea of scale transformations. The $SO(2, 4)$ is more familiar to theorists to be realized as 4D conformal symmetry. Checking a possible matching of the symmetry as isometry of the 6D geometry to that of the conformal symmetry of the 4D space-time part gives interesting conclusions [7]. The symmetry matches to that of the conformal symmetry only on the conformal universe, a hypersurface given by $\eta_{\mathcal{M}\mathcal{N}} X^{\mathcal{M}} X^{\mathcal{N}} = 0$. Translations along the σ - and ρ -coordinate directions within the hypersurface are indeed simple scalings. The quantum world with its characteristic momentum and length scales (quantum scale) is however not scale invariant. we expect the σ - and ρ -coordinate translations to maintain a close connection to scale transformations, if not exactly identical to the latter.

4 Noncommutative space-time

The $SO(2, 4)$ algebra as :

$$\begin{aligned} [M_{\mu\nu}, M_{\lambda\rho}] &= i\hbar (\eta_{\nu\lambda} M_{\mu\rho} - \eta_{\mu\lambda} M_{\nu\rho} + \eta_{\mu\rho} M_{\nu\lambda} - \eta_{\nu\rho} M_{\mu\lambda}) , \\ [M_{\mu\nu}, \hat{P}_\lambda] &= i\hbar (\eta_{\nu\lambda} \hat{P}_\mu - \eta_{\mu\lambda} \hat{P}_\nu) , \\ [M_{\mu\nu}, \hat{X}_\lambda] &= i\hbar (\eta_{\nu\lambda} \hat{X}_\mu - \eta_{\mu\lambda} \hat{X}_\nu) , \\ [\hat{X}_\mu, \hat{X}_\nu] &= \frac{i\hbar}{\kappa^2 c^2} M_{\mu\nu} , \quad [\hat{P}_\mu, \hat{P}_\nu] = -\frac{i\hbar}{\ell^2} M_{\mu\nu} , \\ [\hat{X}_\mu, \hat{P}_\nu] &= -i\hbar \eta_{\mu\nu} \hat{F} , \quad [\hat{X}_\mu, \hat{F}] = \frac{-i\hbar}{\kappa^2 c^2} \hat{P}_\mu , \quad [\hat{P}_\mu, \hat{F}] = \frac{-i\hbar}{\ell^2} \hat{X}_\mu , \end{aligned} \quad (1)$$

($\hbar = \kappa c \ell$). This is to be matched to the standard form

$$[J_{\mathcal{R}\mathcal{S}}, J_{\mathcal{M}\mathcal{N}}] = i\hbar (\eta_{\mathcal{S}\mathcal{M}} J_{\mathcal{R}\mathcal{N}} - \eta_{\mathcal{R}\mathcal{M}} J_{\mathcal{S}\mathcal{N}} + \eta_{\mathcal{R}\mathcal{N}} J_{\mathcal{S}\mathcal{M}} - \eta_{\mathcal{S}\mathcal{N}} J_{\mathcal{R}\mathcal{M}}) , \quad (2)$$

$J_{\mathcal{M}\mathcal{N}} = i\hbar (x_{\mathcal{M}} \partial_{\mathcal{N}} - x_{\mathcal{N}} \partial_{\mathcal{M}})$. We identify

$$\begin{aligned} J_{\mu 4} &\equiv -\kappa c \hat{X}_\mu = i\hbar (x_\mu \partial_4 - x_4 \partial_\mu) , \\ J_{\mu 5} &\equiv -\ell \hat{P}_\mu = i\hbar (x_\mu \partial_5 - x_5 \partial_\mu) , \\ J_{45} &\equiv \kappa c \ell \hat{F} = i\hbar (x_4 \partial_5 - x_5 \partial_4) , \quad J_{\mu\nu} \equiv M_{\mu\nu} . \end{aligned} \quad (3)$$

The result gives an interesting interpretation as suggested by the notation that the generators represent a form of 4D noncommutative geometry. The sets of \hat{X}_μ 's and \hat{P}_μ 's give indeed natural quantum generalizations of the classical x_μ 's and p_μ 's (represented as $i\hbar \partial_\mu$'s here). One can easily check that they do have the right classical limit.

Note that the algebra may also be interpreted as coming from the stabilization of the ‘Poincaré + Heisenberg’ algebra with \hat{F} being the central generator

before deformation. On the quantum world, $-\kappa c \hat{F}$ is rather just the fifth ‘momentum’ component, corresponds to the Beltrami 5-coordinate description. It is reasonable to think that both the Poincaré and Heisenberg algebras must be a part of any ‘quantum space-time’ symmetry. Starting with the ‘Poincaré + Heisenberg’ algebra then gives an alternative clear indication of the need for two step deformations from Einstein relativity. Some may complain that the above interpretation, as the incorporation of the Heisenberg algebra, is rather a phase space symmetry while we set out to look for relativity symmetry as linearly realized on some sort of (configuration) space(-time). The truth is quantum physics is never about configuration space. The latter is next to irrelevant. Quantum mechanics describes only evolution of states, without reference to configurations. Hence, the ‘quantum space-time’ should be more like a quantum phase space of the simple space-time.

We have hence both a 6D classical geometry picture and a 4D noncommutative geometry picture. This may be somewhat in analog to the description of a curved geometry being liable to be described within higher a dimensional flat geometry, such as the 3D picture of a 2D spherical surface. More investigations into the direction may help to provide physicists with a more geometric picture of the generic noncommutative geometry.

5 Final remarks

We have outlined a scheme to think about the physics of the quantum relativity. We consider the subject a sensible and very interesting theoretical endeavor, though conceptually quite radical in comparison to other approaches in the literature. It demands creative but careful thinking about physics beyond the usual framework. The subject is in a very primitive stage, and the challenge ahead is formidable. However, the root of our radical idea of ‘quantum space-time’ as a geometric structure beyond space-time arise from exactly where Einstein’s success as versus Lorentz failure [12] — in putting space into beyond space (space-time). It is amazing to see how much follows from the simple perspective. We hope to be able to keep making small steps in the direction forward.

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